V. *On the Method of determining, from the real Probabilities of Life, the Value of a contingent Reverſion in which Three Lives are involved in the Survivorſhip. By Mr.* William Morgan; *communicated by the Rev.* Richard Price, *D. D. F. R. S.*

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IN a Paper which I had lately the honour of communicating to the Royal Society, reſpecting the method of determining the values of reverſions depending on ſurvivorſhips between two perſons from the real probabilities of life, I obſerved, that the inveſtigation of thoſe caſes in which three lives were involved in the ſurvivorſhip (though attended with much more difficulty) might, however, be effected in a ſimilar manner. The further purſuit of this ſubject has now convinced me that, as it is never ſafe, ſo likewiſe it can never be neceſſary to have recourſe to the *expectations* of life in any caſe; and that the ſolution even of thoſe problems which include three lives is far from being ſo formidable as at firſt ſight it appears to be. I am ſenſible of the impropriety of entering minutely in this place into the vaſt variety of propoſitions which refer to the different orders of ſurvivorſhip between three lives; but as the following problem ſeems to be of conſiderable importance on account of its being applied to the ſolution of many other problems, the demonſtration of it, perhaps, may not be thought an improper addition to my former Paper.

PROBLEM.

Suppoſing the ages of A, B, and C, to be given; to determine, from any table of obſervations, the value of the ſum S payable on the contingency of C's ſurviving B, provided the life of A ſhall be then extinct.

SOLUTION.

Let a repreſent the number of perſons living in the table at the age of A. Let a', a'', a''', a'''', &c. repreſent the decrements of life at the end of the 1ſt, 2d, 3d, 4th, &c. years from the age of A. Let b repreſent the number of perſons living at the age of B, and m, n, o, p, &c. the number of perſons living at the end of the 1ſt, 2d, 3d, 4th, &c. years from the age of B. In like manner let c repreſent the number of perſons living at the age of C, and d, e, f, g, &c. the number of perſons living at the end of the 1ſt, 2d, 3d, 4th, &c. years from the age of C. Let r alſo denote the value of £. 1 increaſed by its intereſt for a year. In order to receive the ſum S in the firſt year, it is neceſſary either that all the three lives ſhall have died in that year, A having died firſt, B next, and C laſt; or that only the two lives A and B ſhall have died (A having died firſt), and that C ſhall have lived to the end of that year. The probability that the three lives ſhall die in the firſt year is $\frac{a' \cdot \overline{b-m} \cdot \overline{c-d}}{abc}$. The probability that they ſhall die in the order above mentioned is $\frac{a' \cdot \overline{b-m} \cdot \overline{c-d}}{6 \cdot abc}$. The probability that both A and B ſhall die in the firſt year is $\frac{a' \cdot \overline{b-m}}{ab}$. Half this fraction, or $\frac{a' \,.\, \overline{b-m}}{2ab}$, is the probability that the death of A ſhall happen before the death of B in this year. The probability that C ſhall ſurvive A and B, reſtrained to the contingency of A's having died firſt, is $\frac{a' \,.\, \overline{b-m} \,.\, d}{2abc}$. The value therefore of the ſum S for the firſt year is $S \times \overline{\frac{a' \,.\, \overline{b-m} \,.\, \overline{c-d}}{6 \,.\, abcr} + \frac{a' \,.\, \overline{b-m} \,.\, d}{2abcr}} = \frac{S}{abcr} \times \overline{\frac{a'bc}{6} - \frac{a'mc}{6} + \frac{a'db}{3} - \frac{a'md}{3}}$ . . . . In the ſecond year the payment of the given ſum will depend on either of four events happening. Firſt, on the contingency of all the three lives dying in that year, A having died firſt, B next, and C laſt. 2dly, On the contingency of B's dying in that year, C's living to the end of it, and A's dying in the firſt year. 3dly, On the contingency of B's dying after A in the ſecond year (both of them having ſurvived the firſt year) and of C's living to the end of that year. 4thly, On the contingency of A's dying in the firſt year, and of B and C's both dying in the ſecond year, B having died firſt. The probability of the firſt contingency is expreſſed by the fraction $\frac{a'' \,.\, \overline{m-n} \,.\, \overline{d-e}}{6 \,.\, abc}$. The probability of the ſecond by the fraction $\frac{a' \,.\, \overline{m-n} \,.\, e}{abc}$. The probability of the third by the fraction $\frac{a'' \,.\, \overline{m-n} \,.\, e}{2 \,.\, abc}$. And the probability of the fourth contingency by the fraction $\frac{a' \,.\, \overline{m-n} \,.\, \overline{d-e}}{2abc}$. Theſe ſeveral fractions, therefore, multiplied into $\frac{S}{r^2}$ will be the value of the given ſum for the ſecond year, and may be eaſily found $= \frac{S}{abcr^2} \times \overline{\frac{a''dm}{6} - \frac{a''dn}{6} + \frac{a''em}{3} - \frac{a''en}{3} + \frac{a'em}{2} - \frac{a'en}{2} + \frac{a'dm}{2} - \frac{a'dn}{2}}$. In like

manner

manner the payment of the given ſum in the third year will depend on the contingency of the ſame number of events as in the ſecond year; that is, it will, firſt, depend on the contingency of all the three lives dying in that year, A having died firſt, B next, and C laſt; 2dly, on the contingency of B's dying in that particular year, C's living to the end of it, and A's dying in the firſt or ſecond years; 3dly, on the contingency of B's dying after A in the third year (both of them having ſurvived the two preceding years), and of C's living to the end of that year; and, 4thly, on the contingency of A's dying in the firſt or ſecond year, and of B and C's both dying in the third year, C having died laſt. Theſe ſeveral contingencies are expreſſed by the reſpective fractions $\frac{a'''.\overline{n-o}.\overline{e-f}}{6.abc}$ . . . $\frac{\overline{a'+a''}.\overline{n-o}.f}{abc}$ . . . $\frac{a''.\overline{n-o}.f}{2abc}$ . . . and $\frac{\overline{a'+a''}.\overline{n-o}.\overline{e-f}}{2abc}$. Conſequently the value of the ſum S for the third year will be $=\frac{S}{abcr^3}\times\overline{\frac{a'''.en}{6}-\frac{a'''.eo}{6}+\frac{a'''.fn}{3}\quad\frac{a'''.fo}{3}+}$ $\overline{\frac{\overline{a'+a''}.fn}{2}-\frac{\overline{a'+a''}.fo}{2}+\frac{\overline{a'+a''}.en}{2}-\frac{\overline{a'+a'}.eo}{2}}$. And by reaſoning in the ſame manner the value of the ſum S for the fourth year may be found $=\frac{S}{abcr^4}+\overline{\frac{a''''.fo}{6}-\frac{a''''.fp}{6}+\frac{a''''.go}{3}-\frac{a''''.gp}{3}+}$ $\overline{\frac{\overline{a'+a''+a'''}.go}{2}-\frac{\overline{a'+a''+a'''}.gp}{2}+\frac{\overline{a'+a''+a'''}.of}{2}-\frac{\overline{a'+a''+a'''}.fp}{2}}$.

If either B or C be the oldeſt of the three lives, theſe ſeries continued to the extremity of that life will expreſs the whole value of the reverſion, which will be $=\frac{S}{6}\times$ $\overline{\frac{a'bc}{abcr}+\frac{a''dm}{abcr^2}+\frac{a'''en}{abcr^3}+\frac{a''''.fo}{abcr^4}}$ + &c. + $\frac{S}{2r}\times\overline{\frac{a'dm}{abcr}+\frac{\overline{a'+a''}.en}{abcr^2}+}$

$$\frac{\overline{a'+a''+a'''}\cdot fo}{abcr^3}+\&c.-\frac{S}{6}\times\overline{\frac{a'mc}{abcr}+\frac{a''na}{abcr^2}+\frac{a'''oe}{abcr^3}+\frac{a''''\cdot pf}{abcr^4}+\&c.}-$$

$$\frac{S}{2r}\times\overline{\frac{a'dn}{abcr}+\frac{\overline{a'+a''}\cdot eo}{abcr^2}+\frac{\overline{a'+a''+a'''}\cdot fp}{abcr^3}}+\&c.+\frac{S}{3}\times\overline{\frac{a'db}{abcr}+\frac{a''\cdot em}{abcr^2}+}$$

$$\overline{\frac{a'''\cdot fn}{abcr^3}+\frac{a''''\cdot go}{abcr^4}}+\&c+\frac{S}{2r}\times\overline{\frac{a'em}{abcr}+\frac{\overline{a'+a''}\cdot fn}{abcr^2}+\frac{\overline{a'+a''+a'''}\cdot go}{abcr^3}}+\&c.$$

$$-\frac{S}{3}\times\overline{\frac{a'md}{abcr}+\frac{a''en}{abcr^2}+\frac{a'''\cdot fo}{abcr^3}+\frac{a''''\cdot gp}{abcr^4}}+\&c.-\frac{S}{2r}\times\overline{\frac{a'en}{abcr}+\frac{\overline{a'+a''}\cdot fo}{abcr^2}+}$$

$$\overline{\frac{\overline{a'+a''+a'''}\cdot gp}{abcr^3}}+\&c.$$

In order to ſum up the firſt and ſecond of theſe ſeries let β repreſent the number of perſons living at the age of F, a perſon one year younger than B, and $\varkappa$ the number of perſons living at the age of K, a perſon one year younger than C. Let FK, BC, AFK, and ABC, repreſent the value of an annuity on the two and three joint lives of F and K, of B and C, of A, F and K, and of A, B and C reſpectively; then will the ſeries $\frac{S}{6}\times\overline{\frac{a'bc}{abcr}+\frac{a''dm}{abcr^2}+\frac{a'''en}{abcr^3}}+\&c.$ be $=\frac{S\cdot\beta\cdot\varkappa}{6\cdot bc}\times$

$$\overline{\frac{bc}{\beta\varkappa r}-\frac{\overline{a-a'}\cdot bc}{a\beta\varkappa r}+\frac{dm}{\beta\varkappa r^2}-\frac{\overline{a-a'-a''}\cdot dm}{a\beta\varkappa r^2}-\frac{a'dm}{a\beta\varkappa r^2}+\frac{en}{\beta\varkappa r^3}-\frac{\overline{a-a'-a''-a'''}\cdot en}{a\beta\varkappa r^3}}$$

$$\overline{-\frac{\overline{a'+a''}\cdot en}{a\beta\varkappa r^3}},\ \&c.=\frac{S\cdot\beta\varkappa}{bc}\times\frac{FK-AFK}{6}\Big(-\frac{S}{6r}\times\overline{\frac{a'dm}{abcr}+\frac{\overline{a'+a''}\cdot en}{abcr^2}},\ \&c.$$

$$=-\frac{S}{6r}\times\overline{\frac{dm}{bcr}-\frac{\overline{a-a'}\cdot dm}{abcr}+\frac{en}{bcr^2}-\frac{\overline{a-a'-a''}\cdot en}{abcr^2}},\ \&c.=\Big)-\frac{S}{6r}\times$$

$\overline{BC-ABC}$. The ſum, therefore, of the two firſt ſeries, or of $\frac{S}{6}\times\overline{\frac{a'bc}{abcr}+\frac{a''\cdot dm}{abcr^2}}+\&c.+\frac{S}{2r}\times\overline{\frac{a'dm}{abcr}+\frac{\overline{a'+a''}\cdot en}{abcr^2}}+\&c.$ is $=\frac{S}{6}\times$ $\frac{\beta\varkappa\cdot\overline{FK-AFK}}{bc}+\frac{S}{3r}\times\overline{BC-ABC}$. Again, let P repreſent a life one year older than B, and let BK, PC, ABK, and APC, repreſent the values of annuities on the two and three joint lives of

B and

B and K, P and C, A, B and K and of A, P and C: then the ſum of the third and fourth ſeries, or of $-\frac{S}{6}\times\overline{\frac{a'mc}{abcr}+\frac{a''nd}{abcr^2}+\&c.}-\frac{S}{2r}\times\overline{\frac{a'dn}{abcr}+\frac{\overline{a'+a''}\,.\,eo}{abcr^2}+\&c.}$ being $=-\frac{S\,.\,\varkappa}{6c}\times\overline{\frac{mc}{\varkappa br}-\frac{\overline{a-a'}\,.\,mc}{ab\varkappa r}+\frac{dn}{b\varkappa r^2}-\frac{\overline{a-a'-a''}\,.\,dn}{ab\varkappa r^2}+\frac{eo}{b\varkappa r^3}-\frac{\overline{a-a'-a''-a'''}\,.\,eo}{ab\varkappa r^3}}$, &c. $\left(-\frac{S}{3r}\times\overline{\frac{a'dn}{abcr}+\frac{\overline{a'+a''}\,.\,eo}{abcr^2}},\ \&c.=\right)-\frac{S\,.\,m}{3br}\times\overline{\frac{a'dn}{acmr}+\frac{\overline{a'+a''}\,.\,eo}{acmr^2}+\&c.}$, will be $=-\frac{S}{6}\times\frac{\varkappa\,.\,\overline{BK-ABK}}{c}-\frac{S}{3r}\times\frac{m\,.\,\overline{PC-APC}}{b}$. The fifth ſeries, $\frac{S}{3}\times\overline{\frac{a'db}{abcr}+\frac{a''em}{abcr^2}}+\&c.$, is $=\frac{S}{3}\times\frac{\beta}{b}\times\overline{\frac{db}{\beta cr}-\frac{\overline{a-a'}\,.\,db}{a\beta cr}+\frac{em}{\beta cr^2}-\frac{\overline{a-a'-a''}\,.\,em}{a\beta cr^2}+\frac{fn}{\beta cr^3}-\frac{\overline{a-a'-a''-a'''}\,.\,fn}{a\beta cr^3}}$, &c. $-\frac{S}{3r}\times\overline{\frac{a'em}{abcr}+\frac{\overline{a'+a''}\,.\,fn}{abcr^2}}$, &c. = (putting FC and AFC for the values of the two and three joint lives of F and C and of A, F, and C) $\frac{S}{3}\times\frac{\beta\,.\,\overline{FC-AFC}}{b}-\frac{S}{3r}\times\overline{\frac{a'em}{abcr}+\frac{\overline{a'+a''}\,.\,fn}{abcr^2}}$, &c. The ſum, therefore, of the fifth and ſixth ſeries, or of $\frac{S}{3}\times\overline{\frac{a'db}{abcr}+\frac{a''em}{abcr^2}},\&c.+\frac{S}{2r}\times\overline{\frac{a'em}{abcr}+\frac{\overline{a'+a''}\,.\,fn}{abcr^2}+\&c.}$ is $=\frac{S}{3}\times\frac{\beta\,.\,\overline{FC-AFC}}{b}+\left(\frac{S}{6r}\times\overline{\frac{a'em}{abcr}+\frac{\overline{a'+a''}\,.\,fn}{abrc^2}},\ \&c.=\right)\frac{S\times d}{6cr}\times\overline{\frac{em}{bdr}-\frac{\overline{a-a'}\,.\,em}{abdr}+\frac{fn}{bdr^2}-\frac{\overline{a-a'-a''}\,.\,fn}{abdr^2}}+\&c.$ If T denote a life one year older than C, and BT, and ABT denote the values of the two and three joint lives of B and T and of A, B, and T, this laſt ſeries will be $=\frac{S}{6r}\times\frac{d\,.\,\overline{BT-ABT}}{c}$, and conſequently the ſum of the fifth and ſixth ſeries will be $=\frac{S}{3}\times\frac{\beta\,.\,\overline{FC-AFC}}{b}+\frac{S}{6r}\times\frac{d\,.\,\overline{BT-ABT}}{c}$. Laſtly, the ſeventh and eighth ſeries, or $-$

$\frac{S}{3}$

$\frac{S}{3}\times\overline{\frac{a'md}{abcr}+\frac{a''en}{abcr^2}+\&c.}-\frac{S}{2r}\times\overline{\frac{a'en}{abcr}+\frac{\overline{a'+a''}.fo}{abcr^2}+\&c.}$ are $=-\frac{S}{3}\times$ $\overline{\frac{dm}{bcr}-\frac{\overline{a-a'}.dm}{abcr}+\frac{en}{bcr^2}-\frac{\overline{a-a'-a''}.en}{abcr^2}+\&c.}-\frac{S}{6r}\times\overline{\frac{a'en}{abcr}+\frac{\overline{a'+a''}.fo}{abcr^2}+\&c.}$ $=-\frac{S}{3}\times\overline{BC-ABC}-\left(\frac{S}{6r}\times\frac{md}{bc}\times\overline{\frac{en}{mdr}-\frac{\overline{a-a'}.en}{amdr}+\frac{of}{mar^2}-\frac{\overline{a-a'-a''}.of}{amdr^2}+\&c.}=\right)\frac{S}{6r}\times\frac{md.\overline{PT-APT}}{bc}$, where PT and APT repreſent the values of the two and three joint lives of P and T, and of A, P, and T. If theſe ſeveral expreſſions be added together, &c. we ſhall at laſt have $\frac{S.x}{6c}\times\overline{\frac{\beta.\overline{FK-AFK}}{b}-\overline{BK-ABK}}$ $+\frac{S.\beta}{3b}\times\overline{FC-AFC}-\frac{S.\overline{r-1}}{3r}\times\overline{BC-ABC}-\frac{S.m}{3br}\times\overline{PC-APC}+$ $\frac{S.d}{6cr}\times\overline{BT-ABT}-\frac{m.\overline{PT-APT}}{b}$, for the value of the ſum S, when either B or C are the oldeſt of the three lives.

In order to determine the value of the reverſion when the life of A is the oldeſt of the three lives, let s, t, u, w, &c. be the number of perſons living at the end of the 1ſt, 2d, 3d, 4th, &c. years from the age of A, and let b', b'', b''', b'''', &c. be the decrements of life at the end of 1, 2, 3, 4, &c. years from the age of B; then, by reaſoning as above, the value of the ſum S for the firſt year will be expreſſed by the ſeries $\frac{S\times b'.\overline{a-s}.\overline{c-d}}{6abcr}+$ $\frac{S\times b'.\overline{a-s}.d}{2abcr}$, for the ſecond year by the ſeries $\frac{S.b''.\overline{s-t}.\overline{d-e}}{6abcr^2}+$ $\frac{S.b''.\overline{s-t}.e}{2abcr^2}+\frac{S.b''\overline{a-s}.e}{abcr^2}+\frac{S.b''.\overline{a-s}.\overline{a-e}}{2abcr^2}$, for the third year by the ſeries $\frac{S.b'''.\overline{t-u}.\overline{e-f}}{6abcr^3}+\frac{S.b'''.\overline{t-u}.f}{2abcr^3}+\frac{S.b'''.\overline{a-t}.f}{abcr^3}+$ $\frac{S.b'''.\overline{a-t}.\overline{e-f}}{2abcr^3}$, and ſo on for the remaining years of A's life.

Theſe ſeveral ſeries may be found $=\frac{S}{abcr}\times\overline{-\frac{ab'c}{3}-\frac{b'cs}{6}-\frac{ab'd}{6}-\frac{b'ds}{3}+}$

$ab'd$

$\overline{\frac{ab'd}{2}+\frac{ab'c}{2}}+\frac{S}{abcr^2}\times\overline{-\frac{sb''d}{3}-\frac{b''dt}{6}-\frac{sb''e}{6}-\frac{be''t}{3}+\frac{ab''e}{2}+\frac{ab''d}{2}}+\frac{S}{abcr^3}\times$ $\overline{-\frac{tb'''e}{3}-\frac{b'''eu}{6}-\frac{tb'''f}{6}-\frac{b'''fu}{3}+\frac{ab'''f}{2}+\frac{ab'''e}{2}}$, &c. &c. Let α repreſent the number of perſons living at the age of H, a perſon one year younger than A; let N denote a perſon one year older than A, and let the ſeveral combinations BN, BNC, AB, &c. denote, as in the former caſe, the values of annuities on the joint lives of B and N, of B, N and C, of A and B, &c.; then by proceeding in the ſame manner as in the foregoing demonſtration the ſeries $\frac{ab'c}{3abcr}+\frac{sb''d}{3abcr^2}+\frac{tb'''e}{3abcr^3}+$ &c. may be found $=\frac{\alpha\varkappa}{3ac}\times\overline{HK-HBK}-\frac{AC-ABC}{3r}$; the ſeries $\frac{b'cs}{6abcr}+\frac{b''dt}{6abcr^2}+\frac{b'''eu}{6abcr^3}+$ &c. $=\frac{\varkappa}{6c}\times\overline{AK-ABK}-\frac{s}{6ar}\times\overline{NC-NBC}$; the ſeries $\frac{ab'd}{6abcr}+\frac{sb''e}{6abcr^2}+\frac{tb'''f}{6abcr^3}+$ &c. $=\frac{\alpha}{6a}\times\overline{HC-HBC}-\frac{d}{6cr}\times\overline{AT-ABT}$; and the ſeries $\frac{b'ds}{3abcr}+\frac{b''et}{3abcr^2}+\frac{b'''fu}{3abcr^3}+$ &c. $=\frac{AC-ABC}{3}-\frac{sd}{3acr}\times\overline{NT-NTB}$. Theſe four ſeries, therefore, ſuppoſing them all to be poſitive quantities are $=\frac{\varkappa}{3c}\times\overline{\frac{\alpha\,.\,\overline{HK-HBK}}{a}+\frac{AK-ABK}{2}}+\frac{\alpha}{6a}\times\overline{HC-HBC}+\frac{\overline{r-1}}{3r}\times\overline{AC-ABC}-\frac{s}{6ar}\times\overline{NC-NBC}-\frac{d}{3cr}\times\overline{\frac{AT-ABT}{2}+\frac{s\,.\,\overline{NT-NBT}}{a}}$. With reſpect to the two remaining ſeries $\frac{b'd}{2bcr}+\frac{b''e}{2bcr^2}+\frac{b'''f}{2bcr^2}+$ &c. . . . and $\frac{b'c}{2bcr}+\frac{b''d}{2bcr^2}+\frac{b'''e}{2bcr^3}+$ &c., theſe, it is evident, are to be continued after the deceaſe of A till the extinction of the joint lives of B and C, and have been already proved in the ſolution of the ſecond problem in my former Paper, to denote the value of the given ſum payable if C ſhould ſurvive B. Let this value be repreſented by **R** and the ſum of the foregoing expreſſions (or $\frac{n}{3c} \times \overline{\frac{\alpha \,.\, \overline{HK - HBK}}{a} + \frac{AK - ABK}{2}} + \frac{\alpha}{6a} \times \overline{HC - HBC}$, &c.) by M, then will the value of the ſum S (when A is the oldeſt of the three lives) be $= S \times \overline{R - M}$. Q. E. D.

If the three lives be equal, the value of the given ſum for the firſt year will be $= \frac{S \,.\, \overline{c-d}|^3}{6 \,.\, c^3 \,.\, r} + \frac{S \,.\, \overline{c-d}|^2 \,.\, d}{2c^3 \,.\, r} = S \times \overline{\frac{1}{6r} + \frac{2a^3}{6c^3r} - \frac{3dd}{6c^2r}}$; the value of the ſame ſum for the ſecond year will be $=$ $\frac{S \,.\, \overline{a-e}|^3}{6c^3r^2} + \frac{S \,.\, \overline{d-e}|^2 \,.\, e}{2c^3r^2} + \frac{S \,.\, \overline{d-e} \,.\, \overline{c-d} \,.\, e}{c^3r^2} + \frac{\overline{d-e}|^2 \,.\, \overline{c-d}}{2c^3r^2} = S \times$ $\overline{\frac{2e^3}{6c^3r^2} - \frac{3ee}{6c^2r^2} - \frac{2d^3}{6c^3r^2} + \frac{3dd}{6c^2r^2}}$; the value for the third year will be $=$ $S \times \overline{\frac{2f^3}{6c^3r^3} - \frac{3ff}{6c^2r^3} - \frac{2e^3}{6c^3r^3} + \frac{3ee}{6c^2r^3}}$, and ſo on for the other years to the extremity of life. Let CC and CCC denote the values of the two equal and three equal joint lives, the ſum of theſe ſeries may then be found $= \frac{S}{6} \times \overline{\frac{1}{r} + \overline{2CCC - 3CC} + \frac{3 \,.\, \overline{CC - 2CCC}}{r}}$ $=$ (ſuppoſing the perpetuity, or $\frac{1}{r-1}$, to be denoted by V) $\frac{S}{6} \times \frac{\overline{r-1}}{r} \times \overline{V - \overline{3 \,.\, CC - 2CCC}}$.

It muſt be here remembered, that from other principles it is well known, that the number of years purchaſe expreſſing the value of an *eſtate* or *perpetual annuity* to be entered upon at the failure of two out of any three equal lives is, "the difference "between three times the values of two equal joint lives, and "twice the values of three equal joint lives ſubtracted from "the perpetuity," or $V - \overline{3CC - 2CCC}$. The value, therefore, of ſuch a reverſion, ſuppoſing it to depend on the failure of the three equal lives in any one particular order, is (ſince there are ſix ſuch orders equally probable) $\frac{1}{6} \times \overline{V - 3CC - 2CCC}$. But it appears, from the correction explained in Dr. PRICE's Treatiſe on Reverſionary Payments, Vol. I. p. 34. that the value of a reverſionary *ſum* is always leſs than the value of an equivalent reverſionary *eſtate* in the proportion of 1 to r. The ſum being S the equivalent eſtate or perpetual annuity is always $S \times \overline{r - 1}$; and conſequently the value of the ſum S depending on the ceaſing of three equal lives in any one particular order and thus determined, is the ſame with that determined by the foregoing inveſtigation, that is, $\frac{S}{6} \times \frac{\overline{r-1}}{r} \times \overline{V - 3CC - 2CC}$. The inveſtigation, therefore, is right, and the correction and inveſtigation demonſtrate one another.

But the foregoing expreſſion for determining the value of the reverſion in this particular caſe is not only obtained immediately from the ſeries, but alſo from the two different rules which have been given for determining the value when the lives are unequal; and hence a proof ariſes of the truth of theſe rules, as well as of the reaſoning upon which they are founded. Thus the firſt rule, ſuppoſing the lives all equal, becomes $\frac{x^2}{c^2} \times \frac{KK - CKK}{6} - \frac{dd}{cc \cdot r} \times \frac{TT - CTT}{6} - \frac{\overline{r-1}}{r} \times \frac{CC - CCC}{3} + \frac{x}{c} \times \frac{CK - CCK}{6} - \frac{d}{cr} \times \frac{CT - CCT}{6}$, and the ſecond rule becomes $\frac{\overline{V - CC} \cdot \overline{r - 1}}{2r} - \frac{x^2}{c^2} \times \frac{KK - CKK}{3} + \frac{dd}{cc \cdot r} \times \frac{TT - CTT}{3} - \frac{r-1}{r} \times \frac{CC - CCC}{3} - \frac{x}{c} \times \frac{CK - CCK}{3} + \frac{d}{cr} \times \frac{CT - CCT}{3}$. Let the value according to the firſt rule be denoted by L, and the ſecond rule will be $= \frac{\overline{r-1} \cdot \overline{V - CC}}{2r} - 2L - \frac{\overline{r-1} \cdot \overline{CC - CCC}}{r} (= L)$. Hence $3L =$

$$\frac{\overline{r-1}\,.\,\overline{V-CC}-2\,.\,\overline{r-1}\,.\,\overline{CC-CCC}}{2r} \text{ and } L=\frac{\overline{r-1}}{6r}\times\overline{V-3CC-2CCC}.$$

Q. E. D.

Were we poſſeſſed of complete tables of the values of annuities on two and three joint lives, the preceding rules would give an eaſy and exact ſolution of this problem in all caſes. But as ſuch tables, computed for every age, would be a work of immenſe difficulty, eſpecially in regard to the values of three joint lives, Mr. SIMPSON's rule for approximating to theſe from the given values of the two joint lives, has hitherto been adopted, and it ſeems upon the whole to anſwer the purpoſe very well. In the preſent problem it is attended with no other inconvenience than increaſing the labour of the computations; for the values of the reverſions derived from it appear in general to be perfectly correct. This is more fully aſcertained by a table which Dr. PRICE has given in his Treatiſe on Reverſionary Payments (Vol. II. Table 37.), of the values of three equal joint lives computed at 4 *per cent.* from the probabilities of life at NORTHAMPTON. By the aſſiſtance of this table, when the lives are of the ſame age, it is evident, from what has been already obſerved, that the exact value of the reverſion may be eaſily obtained. The few following ſpecimens computed from it, and compared with the values of the reverſions deduced from the firſt and ſecond of the preceding rules, demonſtrate the accuracy of thoſe rules: for, notwithſtanding the approximated values of the three joints lives have been uſed in every inſtance in which the rules have been employed, yet the reſults approach ſo near the truth, even in the laſt ſtages of life, when the decrements are moſt irregular, that, though derived from theſe approximations, there can be little doubt of their correctneſs in almoſt every other period of life.

Common age.	Exact value[*] of £. 100. computed from Dr. PRICE's Tables of the values of two and three equal joint lives.	Value of £. 100. computed from the firſt of the foregoing rules, and from Mr. SIMPSON's approximation to the values of three joint lives.	Value of £. 100. computed from the ſecond of the foregoing rules, and from Mr. SIMPSON's approximation to the values of three joint lives.
70	12.000	12.005	12.000
75	12.944	12.943	12.943
80	13.840	13.810	13.880
85	14.450[†]	14.670	14.340

Mr. DODSON[‡], and Mr. SIMPSON[§], are the only writers who have ſolved, or rather who have approximated to the ſolution of this problem. But the former, by deducing his rules immediately from a wrong hypotheſis, having rendered moſt of them (eſpecially thoſe in which three lives are concerned) of no uſe, it will be unneceſſary to take notice of what he has done on the ſubject. With regard to the latter, whoſe rule is not only the ſole guide for determining the value of this reverſion, but alſo the ſource from which a great variety of other problems are ſolved, perhaps it may not be improper to examine how far his ſolution is to be depended *upon; and the following* examples have therefore been computed for this purpoſe.

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[*] That is, *one-ſixth* part of the whole reverſion.

[†] The ſeveral reverſions in this column, when computed from SIMPSON's approximation to the values of the three joint lives, are 12.012, 12.933, 13.847, and 14.803 reſpectively; which upon the whole differing nearly as much from the real values as thoſe in the two other columns afford a convincing proof, that the very ſmall deviation from the truth in theſe latter values proceeds not from any inaccuracy in the rules themſelves, but ſolely from having uſed the *approximated* inſtead of the *real* values of the three joint lives. And this alſo will account for the difference in the values by the firſt and ſecond rules. Were thoſe values computed from tables which give the correct values of two and three joint lives at all ages, they would come out exactly the ſame. In the two firſt examples, where the values by one rule are true, it appears, that the values by the other rule are equally ſo. In the two laſt examples, where the values are not quite ſo accurate, it may be obſerved, that they differ as much in exceſs by one rule as they do in defect by the other; which muſt in general be the caſe from the very nature of thoſe rules; for if L (or the value by the firſt rule) be greater than the truth, the difference between $\frac{\overline{r-1}\,.\,\overline{V-CC}}{2r}$ and 2L (or the value by the ſecond rule) muſt be leſs than the truth; and, on the contrary, if L be leſs, this difference will be greater than the truth.

[‡] See DODSON's Mathematical Repoſitory, Vol. III. Queſtions 42, 43, &c.

[§] See SIMPSON's Select Exerciſes, Prob. 38.

TABLE I.

Ages of			Value, by SIMPSON's rule, of £. 100. payable on the contingent reverſion ſpecified in this problem, when either C or B are eldeſt, according to the *Northampton* Table and at 4 *per cent.*	True value of the ſame reverſion computed from the firſt rule in the foregoing ſolution.
C.	B.	A.		
80	70	40	1.926	1.179
75	65	25	1.873	1.032
65	50	15	2.090	1.690
70	80	40	6.615	6.117
50	65	15	5.580	3.879
78	78	20	2.583	1.982
45	60	12	5.571	4.133
60	45	12	2.292	1.686

TABLE II.

Ages.			Value of the ſame reverſion by SIMPSON's rule, when A is the oldeſt of the three lives.	True value of the ſame reverſion by the ſecond rule in the foregoing ſolution.
C.	B.	A.		
24	65	75	34.636	31.792
65	24	75	6.305	7.895
49	9	69	7.351	5.960
18	78	78	37.554	33.019

TABLE III.

Common Age.	Value of the ſame reverſion by SIMPSON's rule, when the ages of the three lives are equal.	True value of the ſame reverſion.
70	13.20	12.00
75	14.98	12.94
80	16.58	13.84
85	17.86	14.45

By comparing the values in the preceding tables, Mr. SIMPSON's rule appears in almoſt every inſtance to be exceedingly incorrect. Even when the lives are equal (in which caſe it might have been expected to be ſufficiently accurate) it ſeems to deviate, in old age at leaſt, ſo widely from the truth as to be unfit for uſe. When C or B are eldeſt (which, however, is a caſe that does not often occur), the reſults ſometimes exceed the truth *one-half*, and generally by more than *one-third* of the real value. When A is the oldeſt of the three lives (which is the moſt common caſe) theſe reſults are erroneous in nearly an equal degree. Nay, in ſome caſes, Mr. SIMPSON's rule is not only wrong but abſurd. Thus, in the laſt example in the ſecond table, the value of £. 100. payable on the contingency of C aged 18 ſurviving B aged 78 after A aged 78, is by this rule = £. 37.554. The value, therefore, of the ſame ſum on the contingency of C's ſurviving A after B is alſo £. 37.554. Hence the value of £. 100. on the contingency of C's ſurviving A and B (without the reſtriction of one dying before the other) is $2 \times 37.554 =$ £. 75.108 *. By another rule of Mr. SIMPSON †, the value of £. 100, on the contingency of C's ſurviving B only, is no more than £. 74*. Now it is ſelf-evident, that this latter value, inſtead of being *leſs*, ought to have been *greater* than the former, inaſmuch as the probability of C's ſurviving only one life muſt be greater than that of his ſurviving two lives.

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* See SIMPSON's Select Exerciſes, Prob. 39. † Ibid. Prob. 32.

Many additional inſtances might be produced in which this rule, being made the baſis upon which the ſolutions of other problems are founded, leads to concluſions equally erroneous. But theſe enquiries would be improper here; and I ſhall only obſerve, that had the foregoing examples been computed from the SWEDEN or LONDON, inſtead of the NORTHAMPTON Table, this rule would have appeared to be ſtill more incorrect than it does from thoſe computations.

When Mr. SIMPSON wrote his Select Exerciſes, he was in a great meaſure obliged to have recourſe to DE MOIVRE's hypotheſis, for want of thoſe excellent tables of the real probabilities of life, and alſo of the values of ſingle and joint lives which have been ſince publiſhed. Had he been poſſeſſed of theſe, it is moſt likely that his ſuperior abilities would have directed him to a more accurate method of inveſtigation. At preſent there can be no juſt reaſon for ever recurring to this wretched hypotheſis. The ſolutions of all caſes of two and even of three lives may be effected without much difficulty from principles ſtrictly true. But I muſt here take my leave of this ſubject, hoping that its importance may engage other mathematicians to the further proſecution of it.

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* The true values are £. 66.038. and £. 74.884. reſpectively.

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